

# On the Dynamical Invariants and the Geometric Phases for a General Spin System in a Changing Magnetic Field

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## Abstract

We consider a class of general spin Hamiltonians of the form  $H_s(t) = H_0(t) + H'(t)$  where  $H_0(t)$  and  $H'(t)$  describe the dipole interaction of the spins with an arbitrary time-dependent magnetic field and the internal interaction of the spins, respectively. We show that if  $H'(t)$  is rotationally invariant, then  $H_s(t)$  admits the same dynamical invariant as  $H_0(t)$ . A direct application of this observation is a straightforward re-derivation of the results of Yan et al [Phys. Lett. A **251**(1999) 289 and **259** (1999) 207] on the Heisenberg spin system in a changing magnetic field.

## 1 Introduction

In Ref. [1], the authors construct a dynamical invariant [2] for the Heisenberg spin system in a changing magnetic field.<sup>1</sup> This invariant involves two auxiliary functions that satisfy a system of coupled first order differential equations. These same differential equations arise in the construction of a dynamical invariant for the dipole interaction of a single spin in a changing magnetic field. This observation together with the more recent results of Ref. [4] on the characterization of the quantum systems admitting the same dynamical invariant are the main motivation for the present study.

The Hamiltonian  $H_{\text{Heisenberg}}$  for the Heisenberg spin system in a changing magnetic field  $\vec{B}(t)$  is a special case of the spin Hamiltonians of the form

$$H_s(t) = H_0(t) + H'(t), \quad (1)$$

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<sup>1</sup>See also [3].

where  $H_0(t)$  is the dipole interaction Hamiltonian given by

$$H_0(t) := \vec{B}(t) \cdot \sum_{i=1}^N \vec{S}_i = \sum_{i=1}^N \sum_{\alpha=1}^3 B^\alpha(t) S_i^\alpha, \quad (2)$$

$\vec{S}_i = (S_i^1, S_i^2, S_i^3)$  is the spin operator for the  $i$ -th particle,  $N$  is the number of particles, and  $H'(t)$  is the Hamiltonian corresponding to the internal spin interaction of the system. For  $H_{\text{Heisenberg}}(t)$ ,  $H'(t)$  is a time-independent Hamiltonian given by

$$H'_{\text{Heisenberg}} = -AH_1, \quad H_1 := \sum_{i_1, i_2} \sum_{\alpha_1, \alpha_2} Q_{i_1, i_2}^{\alpha_1 \alpha_2} S_{i_1}^{\alpha_1} S_{i_2}^{\alpha_2}, \quad (3)$$

$$Q_{i_1 i_2}^{\alpha_1 \alpha_2} := \begin{cases} \delta_{\alpha_1, \alpha_2} & \text{if } i_1 \text{ and } i_2 \text{ label particles that are nearest neighbors} \\ 0 & \text{otherwise,} \end{cases} \quad (4)$$

where  $A$  is a constant and  $\delta_{a,b}$  denotes the Kronecker delta function.

The purpose of this article is to show that any Hamiltonian of the form (1) admits a dynamical invariant that is also a dynamical invariant of the dipole Hamiltonian (2) provided that  $H'(t)$  has rotational invariance.

First, we recall that by definition a dynamical invariant  $I(t)$  for a Hamiltonian  $H(t)$  is a (nontrivial) solution of the Liouville-von-Neumann equation:

$$\frac{d}{dt} I(t) = i[I(t), H(t)], \quad (5)$$

and that any dynamical invariant satisfies

$$I(t) = U(t)I(0)U^\dagger(t), \quad (6)$$

where  $U(t) = \mathcal{T}e^{-i \int_0^t H(t')dt'}$  is the evolution operator generated by the Hamiltonian  $H(t)$  and  $\mathcal{T}$  is the time-ordering operator.

It is well-known [5, 6] that one can construct a dynamical invariant  $I_i(t)$  for a spin in a changing magnetic field using the ansatz  $I_i(t) = \vec{R}(t) \cdot \vec{S}_i$ . Substituting this expression for  $I(t)$  in Eq. (5) and using the relation  $H(t) = \vec{B}(t) \cdot \vec{S}_i$  for the Hamiltonian and the  $su(2)$  algebra:  $[S_i^\alpha, S_i^\beta] = i \sum_{\gamma=1}^3 \epsilon_{\alpha\beta\gamma} S_i^\gamma$ , one obtains a system of first order differential equations for the functions  $R^\alpha(t)$ , [5, 6]. This invariant can also be expressed in the form  $I_i(t) = W_i(t)S_i^3W_i^\dagger(t)$  where  $W_i(t)$  belongs to the (spin representation of the) group  $SU(2)$  generated by  $S_i^\alpha$  and satisfies  $W_i(0) = 1$ . It is also not difficult to see that in view of Eq. (6),

the evolution operator  $U_i(t)$  generated by  $H(t)$  satisfies  $U_i(t) = W_i(t)Z_i(t)$  where  $Z_i(t)$  is a unitary operator commuting with  $S_i^3$ , [4]. These results can be directly employed for the Hamiltonian  $H_0(t)$  of Eq. (2). Specifically, this Hamiltonian admits the invariant

$$I(t) = \sum_{i=1}^N I_i(t) = \vec{R}(t) \cdot \sum_{i=1}^N \vec{S}_i = W_0(t) \left( \sum_{i=1}^N S_i^3 \right) W_0^\dagger(t) = U_0(t) \left( \sum_{i=1}^N S_i^3 \right) U_0^\dagger(t) = U_0(t) I(0) U_0^\dagger, \quad (7)$$

where  $W_0(t) = \prod_{i=1}^N W_i(t)$  and  $U_0(t) = \prod_{i=1}^N U_i(t)$  and

$$I(0) = \sum_{i=1}^N S_i^3. \quad (8)$$

Note that because  $S_i^\alpha$  with different values of  $i$  commute,  $[W_i(t), W_j(t)] = [U_i(t), U_j(t)] = 0$ . Furthermore, we have  $U_i(t) = e^{iM_i(t)}$ ,  $M_i(t) := \vec{\rho}(t) \cdot \vec{S}_i$ , and  $U_0(t) = e^{iM(t)}$  where  $\vec{\rho}(t)$  are determined in terms of  $\vec{R}(t)$  and  $M(t) := \vec{\rho}(t) \cdot \sum_{i=1}^N \vec{S}_i$ .

Now, consider a spin Hamiltonian of the form (1) and suppose that  $H'(t)$  is rotationally invariant, i.e., for all  $\alpha \in \{1, 2, 3\}$ ,

$$[H'(t), \sum_{i=1}^N S_i^\alpha] = 0. \quad (9)$$

Then

$$[H_0(t), H'(t)] = 0, \quad (10)$$

$$[I(t), H'(t)] = 0. \quad (11)$$

In view of Eqs. (1) and (11) and the fact that  $I(t)$  is a dynamical invariant for  $H_0(t)$ , we have

$$\frac{d}{dt} I(t) = i[I(t), H_0(t)] = i[I(t), H_0(t) + H'(t)] = i[I(t), H_s(t)].$$

Hence  $I(t)$  is also a dynamical invariant for the Hamiltonian  $H_s(t)$ . Furthermore, according to Eqs. (1) and (10), the evolution operator  $U_s(t)$  generated by  $H_s(t)$  is the product of those generated by  $H_0(t)$  and  $H'(t)$ , i.e.,

$$U_s(t) = U_0(t)U'(t), \quad (12)$$

where  $U'(t) = \mathcal{T} e^{-i \int_0^t H'(t') dt'}$ .

As we mentioned earlier, the Heisenberg spin Hamiltonian  $H_{\text{Heisenberg}}$  considered in Refs. [1, 3] is a special case of  $H_s(t)$ . It can also be easily checked that  $H'_{\text{Heisenberg}}$  is rotationally invariant. Hence  $H_{\text{Heisenberg}}$  also admits the invariant  $I(t)$  of Eq. (7). The invariant constructed in Ref. [1] differs from the invariant (7) by a constant term that commutes with the Hamiltonian. Therefore, this term drops from both sides of the defining equation (5). The only effect of this additional term is to change the degeneracy of the eigenvalues of the invariant. In fact, the most general dynamical invariant for the Hamiltonian  $H_s(t)$  (and in particular for  $H_{\text{Heisenberg}}$ ) is given by

$$I_s(t) = U_s(t)I(0)U_s^\dagger(t) = U_0(t)U'(t)I(0)U'^\dagger(t)U_0^\dagger(t),$$

where  $I(0)$  is a constant Hermitian operator. The invariant (7) corresponds to the choice (8) for  $I(0)$ . The invariant constructed in Ref. [1] corresponds to the choice

$$I(0) = \sum_{i=1}^N S_i^3 + H_1. \quad (13)$$

As mentioned in Ref. [1], the eigenvectors of the invariant corresponding to (13) are not known. This means that an explicit solution of the time-dependent Schrödinger equation using this invariant is not possible. Unlike this invariant the eigenvectors of the invariant (7) are easily calculated; they are (tensor) products of the eigenvectors of  $I_i(t)$ , i.e.,  $W_i(t)|n_i\rangle$  where  $|n_i\rangle$  are the eigenvectors of  $S_i^3$  with eigenvalue  $n_i = \pm 1/2$ .

We wish to conclude this article with the following remarks.

1. Because it is the dynamical invariants that determine the geometric phases [7, 6], the geometric phases obtained for the Hamiltonian  $H_s(t)$  coincide with those of the dipole Hamiltonian  $H_0(t)$ . This is the reason why the expression obtained in Ref. [1] for the geometric phases of a Heisenberg spin system in a changing magnetic field is essentially the same as the one for the geometric phases of a single spin in the same magnetic field.
2. We can construct more general internal interaction Hamiltonians  $H'(t)$  that are rotationally invariant and thus the corresponding total Hamiltonian  $H_s(t)$  admits the same

dynamical invariant as  $H_0(t)$ . For example, we can set

$$H'(t) = \sum_n \lambda_n(t) H_n, \quad (14)$$

$$H_n := \sum_{i_1, \dots, i_{2n}=1}^N \sum_{\alpha_1, \dots, \alpha_{2n}=1}^3 Q_{i_1 \dots i_{2n}}^{\alpha_1 \dots \alpha_{2n}} S_{i_1}^{\alpha_1} \dots S_{i_{2n}}^{\alpha_{2n}}, \quad (15)$$

where  $n$  takes positive integer values,  $\lambda_n$  are real-valued functions of  $t$ , and  $Q_{i_1 \dots i_{2n}}^{\alpha_1 \dots \alpha_{2n}}$  are real coupling constants satisfying certain symmetry conditions. In order to state these conditions, we introduce the following abbreviated notation

$$\tilde{Q}_{i_j i_k}^{\mu, \nu} := Q_{i_1 \dots i_j \dots i_k \dots i_{2n}}^{\alpha_1 \dots \alpha_j \dots \alpha_k \dots \alpha_{2n}} \quad \text{with} \quad \alpha_j = \mu, \quad \alpha_k = \nu.$$

Then the above mentioned conditions take the form

$$\text{for } i_j = i_k : \quad \tilde{Q}_{i_j i_k}^{\mu\nu} = \tilde{Q}_{i_j i_k}^{\nu\mu}, \quad (16)$$

$$\begin{aligned} \text{for } i_j \neq i_k : \quad & \delta_{\mu, \gamma} \left( \tilde{Q}_{i_j i_k}^{\beta\nu} + \tilde{Q}_{i_k i_j}^{\nu\beta} \right) + \delta_{\nu, \gamma} \left( \tilde{Q}_{i_j i_k}^{\mu\beta} + \tilde{Q}_{i_k i_j}^{\beta\mu} \right) - \\ & \delta_{\mu, \beta} \left( \tilde{Q}_{i_j i_k}^{\gamma\nu} + \tilde{Q}_{i_k i_j}^{\nu\gamma} \right) - \delta_{\nu, \beta} \left( \tilde{Q}_{i_j i_k}^{\mu\gamma} + \tilde{Q}_{i_k i_j}^{\gamma\mu} \right) = 0. \end{aligned} \quad (17)$$

These relations must be satisfied for all possible values of the labels  $j, k, i_j, i_k, \beta, \gamma, \mu$ , and  $\nu$ . They follow from the requirement that  $H_1$  commutes with the total spin operators  $\sum_{i=1}^N S_i^\alpha$  and the identities

$$\begin{aligned} S_i^\alpha S_i^\beta &= \frac{1}{4} \delta_{\alpha, \beta} + \frac{i}{2} \sum_{\gamma=1}^3 \epsilon_{\alpha\beta\gamma} S_i^\gamma, \\ S_i^\alpha S_j^\beta &= S_j^\beta S_i^\alpha \quad \text{for } i \neq j. \end{aligned}$$

It is not difficult to generalize the conditions obtained for  $H_1$  to  $H_n$ .

Perhaps the simplest nontrivial example that fulfils conditions (16) and (17) is  $Q_{i_1 i_2}^{\alpha_1 \alpha_2}$  of Eq. (4).

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